Multiple defaults and contagion risks

Ying Jiao*

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Abstract

We study multiple defaults where the global market information is modelled as progressive enlargement of filtrations. We shall provide a general pricing formula by establishing a relationship between the enlarged filtration and the reference default-free filtration in the random measure framework. On each default scenario, the formula can be interpreted as a Radon-Nikodym derivative of random measures. The contagion risks are studied in the multi-defaults setting where we consider the optimal investment problem in a contagion risk model and show that the optimization can be effectuated in a recursive manner with respect to the default-free filtration.

1 Introduction

The contagion credit risk analysis with multiple default events is an important issue for evaluating the credit derivatives and for the risk management facing the financial crisis. Compared to the single credit name studies, there are several difficulties in the multi-defaults context. Generally speaking, the global market information containing all defaults information is modelled as a recursive enlargement of filtrations of all default times with respect to a default-free reference filtration. To obtain the value process of a credit-sensitive claim, one needs to consider the conditional expectation of its payoff function with respect to the global market filtration. The mathematical formulation and computations are in general complicated considering all possible default scenarios and the enlarged filtration. Furthermore, the modelling of correlation structures of default times in a dynamic manner is a challenging subject, in particular, when it concerns how to take into consideration the impact of one default event on the remaining names.

In the literature, there are mainly two approaches – bottom up and top down – to model multiple default events. In the first approach, one is interested in the probability distributions of each individual default and in their correlations, often using copula functions. The second approach concentrates directly on the cumulative losses distributions, which allows to reduce the complexity of the problem. However, the correlation structure

*Laboratoire de probabilités et modèles aléatoires, Université Paris 7, jiao@math.jussieu.fr.

between the default times is not straightforward in the top-down models. Recently, a new approach has been proposed to study the successive default events (see [5], also [3]), which provides an intermediary point of view between the above two approaches. The main ideas of [5] are two-folded. On one hand, the default scenarios are largely reduced and we can thoroughly analyze the impact of each default event on the following ones; on the other hand, the computations are decomposed on each default scenario and hence concern only the default-free reference filtration. One key hypothesis is that the family of default times admits a density with respect to the reference filtration. The density hypothesis is a standard one in the enlargement of filtrations (see for example [7] for the initial enlargement of filtration). In the credit risk analysis, the density hypothesis has been adopted in [4] for analyzing what goes on after a default event, it has also been proved to be useful in the recursive before-default and after-default extensions.

Inspired by the second idea mentioned previously in [5], we study the non-ordered multiple defaults by establishing a formal relationship between the global market filtration and the default-free reference filtration. We shall adopt the framework of random measures, which will provide us convenient and concise notations. One can find a detailed introduction to random measures in [8, Chap II] and in the monograph [2]. Similar notions have also been used in the filtering problems (see e.g. [12] and [15]). The main advantage of introducing such a general framework is that we can treat the multiple defaults case in a coherent way as in the single name case. Another consequence is that we can remove the density hypothesis. In the case where explicit results are needed and where the density hypothesis is assumed, we recover a result in [5], to which we refer for more detailed discussions.

As applications, we are interested in the pricing with multiple defaults and in the contagion risks. The important idea in both cases, as mentioned above, is to find a suitable decomposition of the problem on each default scenario, so that the analysis will only concern the default-free filtration. We shall present a general pricing formula, which gives the value process with respect to the global market information. On each default scenario, the formula can be interpreted as a Radon-Nikodym derivative of random measures. This result can be applied to credit portfolio derivatives and also to contingent claims subjected to contagion default risks.

The contagion credit risk during the financial crisis is an important subject which needs to be taken into consideration. Notably, one default event can have significant impact on remaining firms on the market and may become the potential cause of other defaults, called the contagious defaults. We shall present a contagion risk model to describe this phenomenon, where each asset is influenced by the default risks of an underlying portfolio and has a jump on its value at every default time. We consider furthermore an investment portfolio containing such assets and study the optimal investment strategy. We show that the global optimization problem is equivalent to a family of recursive optimization problems with respect to the default-free filtration.

The paper is organized as follows. We present in Section 2 the mathematical framework of random measures. A general pricing result concerning multiple defaults is deduced using

random measures in Section 3, and is applied to credit portfolio derivatives. The Section 4 is devoted to the analysis on contagion risks. We firstly present a multi-defaults contagion model, and then study the optimal investment problems in the presence of contagion risks.

2 Random measure framework

2.1 Preliminaries

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space equipped with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ be a family of random times taking values in $\mathbb{R}_+^n$, considered as $\mathbb{R}_+^{\{1,\cdots,n\}}$, representing the family of default times. Denote by Θ the index set $\{1, \cdots, n\}$. We suppose that τ_i $(i \in \Theta)$ are strictly positive and finite, and that $\tau_i \neq \tau_j$, a.s. for $i \neq j$ $(j \in \Theta)$. Let $\mathbb{D}^i = (\mathcal{D}_t^i)_{t\geq 0}$ be the smallest right-continuous filtration such that τ_i is a $\mathbb{D}^i$-stopping time. More precisely, $\mathcal{D}_t^i := \bigcap_{\varepsilon>0} \sigma(\tau_i \wedge (t+\varepsilon))$. Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the progressive enlargement of $\mathbb{F}$ by the default filtrations, namely, $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \vee \cdots \vee \mathbb{D}^n$.

For any $I \subset \Theta$, let $\tau_I = (\tau_i)_{i\in I}$, which is a random variable valued in $\mathbb{R}_+^I$. For $t \in \mathbb{R}_+$, the notation A_t^I denotes the event

$$A_t^I := \Big(\bigcap_{i\in I}\{\tau_i \leq t\}\Big) \cap \Big(\bigcap_{i\notin I}\{\tau_i > t\}\Big).$$

The events $(A_t^I)_{I\subset\Theta}$ describe all default scenarios at time t. Note that Ω is the disjoint union of $(A_t^I)_{I\subset\Theta}$.

The following lemma is an extension of a classical result on progressive enlargement of filtrations with one default name.

Lemma 2.1 *For $t \in \mathbb{R}_+$, any $\mathcal{G}_t$-measurable random variable Y_t can be written in the decomposed form*

$$Y_t = \sum_{I\subset\Theta} 1\!\!1_{A_t^I} Y_t^I(\tau_I) \tag{1}$$

where $Y_t^I(\cdot)$ is a $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable function on $\Omega \times \mathbb{R}_+^I$, $\mathcal{B}(\mathbb{R}_+^I)$ being the Borel σ-algebra.

Proof. By definition, for any $t \in \mathbb{R}_+$ and any integer $m > 0$, the random variable Y_t is $\mathcal{F}_t \vee \sigma(\boldsymbol{\tau} \wedge (t+1/m))$-measurable. Hence there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function F_m such that $Y_t(\omega) = F_m(\omega, \boldsymbol{\tau} \wedge (t+1/m))$. For $I \subset \Theta$,

$$1\!\!1_{A_t^I} Y_t = 1\!\!1_{\{\tau_I\leq t,\, \tau_{I^c}>t\}} F_m\Big(\omega, \boldsymbol{\tau} \wedge (t+\frac{1}{m})\Big).$$

So, for fixed ω, one has

$$\mathbb{1}_{A_t^I}(\omega) Y_t(\omega) = \mathbb{1}_{A_t^I}(\omega) F_m\Big(\omega, \tau_I(\omega), \big(t+\frac{1}{m}\big)_{I^c}\Big)$$

when m is large enough. Let

$$Y_t^I(s_I) := \limsup_{m\to\infty} F_m(\omega, \boldsymbol{x}^{(m)})$$

where $\boldsymbol{x}^{(m)} = (x_1^{(m)}, \cdots, x_n^{(m)})$ is defined as $x_i^{(m)} := s_i$ if $i \in I$ and $x_i^{(m)} := t+1/m$ if $i \notin I$. Then one has $\mathbb{1}_{A_t^I} Y_t = \mathbb{1}_{A_t^I} Y_t^I(\tau_I)$. □

The following variant of Lemma 2.1 will be useful further on.

Lemma 2.2 *Any $\mathcal{G}_{t-}$-measurable random variable Y_t can be written as*

$$Y_t = \sum_{I\subset\Theta} \mathbb{1}_{A_{t-}^I} Y_t^I(\tau_I),$$

where

$$A_{t-}^I := \Big(\bigcap_{i\in I}\{\tau_i < t\}\Big) \cap \Big(\bigcap_{i\notin I}\{\tau_i \geq t\}\Big),$$

and $Y_t^I(\cdot)$ is $\mathcal{F}_{t-}\otimes\mathcal{B}(\mathbb{R}_+^I)$-measurable.

Proof. For any $t > 0$, $\mathcal{G}_{t-} = \bigcup_{\varepsilon>0} \mathcal{F}_{t-\varepsilon} \vee \sigma(\boldsymbol{\tau}\wedge(t-\varepsilon))$. So there exists an integer $m > 0$ such that Y_t is $\mathcal{F}_{t-}\otimes\sigma(\boldsymbol{\tau}\wedge(t-1/m))$-measurable. Hence Y_t can be written as $F_m(\omega, \boldsymbol{\tau}\wedge(t-1/m))$ where F_m is an $\mathcal{F}_{t-}\otimes\mathcal{B}(\mathbb{R}_+^n)$-measurable function. Then we can complete the proof by a similar argument as for Lemma 2.1. □

Remark 2.3 In the Lemmas 2.1 and 2.2, if the random variable Y_t is positive (resp. bounded), then $Y_t^I(\cdot)$ can be chosen to be positive (resp. bounded).

2.2 Random measures

Definition 2.4 Let $\mu^{\boldsymbol{\tau}}$ be the measure on $(\Omega\times\mathbb{R}_+^n, \mathcal{F}_\infty\otimes\mathcal{B}(\mathbb{R}_+^n))$ such that for any positive and $\mathcal{F}_\infty\otimes\mathcal{B}(\mathbb{R}_+^n)$-measurable function $h_\infty(\cdot)$,

$$\int h_\infty(\boldsymbol{s})\mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}) = \mathbb{E}[h_\infty(\boldsymbol{\tau})], \tag{2}$$

where $\mathcal{F}_\infty = \bigcup_{t\geq 0}\mathcal{F}_t$ and $\boldsymbol{s} = (s_1, \cdots, s_n)$.

The measure $\mu^{\boldsymbol{\tau}}$ can be considered as a transition kernel from $(\Omega, \mathcal{F}_\infty)$ to $(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n))$ whose marginal on Ω coincides with $\mathbb{P}$. It can also be considered as the conditional law of $\boldsymbol{\tau}$ on $\mathcal{F}_\infty$. We give below an example of $\mu^{\boldsymbol{\tau}}$ using the copula model in [14].

Example 2.5 (Schönbucher and Schubert) For any $i \in \Theta$, define the default time by $\tau_i = \inf\{t : \Lambda_t^i \geq U_i\}$ where Λ^i is an continuous increasing $\mathbb{F}$-adapted process and U_i is an exponential distributed random variable independent of $\mathcal{F}_\infty$. The conditional survival probability is $u_t^i = \mathbb{P}(\tau_i > t|\mathcal{F}_\infty) = \exp(-\Lambda_t^i)$. Note that H-hypothesis holds in this model, that is, $\mathbb{P}(\tau_i > t|\mathcal{F}_t) = \mathbb{P}(\tau_i > t|\mathcal{F}_\infty)$. The construction of joint survival distribution in [14] is by introducing a copula function $C : \mathbb{R}_+^n \to \mathbb{R}_+$ such that

$$\mathbb{P}(\boldsymbol{\tau} > \boldsymbol{s}|\mathcal{F}_\infty) = \mathbb{P}(\tau_1 > s_1, \cdots, \tau_n > s_n|\mathcal{F}_\infty) = C(u_{s_1}^1, \cdots, u_{s_n}^n).$$

Then for any positive and $\mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function $h_\infty(.)$, one has

$$\mathbb{E}[h_\infty(\boldsymbol{\tau})] = \int h_\infty(\boldsymbol{s})\mu^{\boldsymbol{\tau}}(d\omega, ds) = \mathbb{E}\Big[\int_{\mathbb{R}_+^n} h_\infty(\boldsymbol{s})(-1)^n d_{s_1}\cdots d_{s_n} C(u_{s_1}^1, \cdots, u_{s_n}^n)\Big]$$

where $d_{s_1}\cdots d_{s_n} C(u_{s_1}^1, \cdots, u_{s_n}^n)$ is an n-dimensional Lebesgue-Stieltjes measure associated to $C(u_{s_1}^1, \cdots, u_{s_n}^n)$.

Classically the random measure is a straightforward extension of the notions of increasing processes and their compensators, see [8, Chap II] for details. Here the random measure $\mu^{\boldsymbol{\tau}}$ is useful to define auxiliary measures on suitable σ-fields. For $t \geq 0$, let $\mu_t^{\boldsymbol{\tau}}$ be the restriction of $\mu^{\boldsymbol{\tau}}$ on $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$. It represents the conditional law of $\boldsymbol{\tau}$ on $\mathcal{F}_t$. For this reason, we also write $\mu_t^{\boldsymbol{\tau}}$ as $\mathbb{E}[\mu^{\boldsymbol{\tau}}|\mathcal{F}_t]$. If $h_t(\boldsymbol{s})$ is a positive $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function, then

$$\mathbb{E}[h_t(\boldsymbol{\tau})] = \int h_t(\boldsymbol{s})\mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}) = \int h_t(\boldsymbol{s})\mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}).$$

For $I \subset \{1, \cdots, n\}$, let μ_t^I be the measure on $(\Omega \times \mathbb{R}_+^I, \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I))$ which is a partial marginal measure of $\mu_t^{\boldsymbol{\tau}}$ such that for any positive and $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable function $h_t(s_I)$, $s_I = (s_i)_{i\in I}$, one has

$$\int_{\Omega\times\mathbb{R}_+^I} h_t(s_I)\mu_t^I(d\omega, ds_I) = \int_\Omega \int_{\mathbb{R}_+^I \times]t,\infty[^{I^c}} h_t(s_I)\mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}). \tag{3}$$

This relation can also be written as

$$\mu_t^I(d\omega, ds_I) = \int_{]t,\infty[^{I^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}). \tag{4}$$

For $T \geq t$, $I \subset \{1, \cdots, n\}$ and $Y_T(\cdot)$ which is positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable, we define $\mu_t^{Y_T,I}$ as the weighted marginal measure on $(\Omega \times \mathbb{R}_+^I, \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I))$ such that

$$\int_{\Omega\times\mathbb{R}_+^I} h_t(s_I)\mu_t^{Y_T,I}(d\omega, ds_I) = \int_\Omega \int_{\mathbb{R}_+^I \times]t,\infty[^{I^c}} h_t(s_I)Y_T(\boldsymbol{s})\mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}). \tag{5}$$

Similarly, we write $\mu_t^{Y_T,I}$ as

$$\mu_t^{Y_T,I}(d\omega, ds_I) = \int_{]t,\infty[^{I^c}} \mathbb{E}[Y_T(\boldsymbol{s})\mu^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s}), \tag{6}$$

where $E[Y_T(\boldsymbol{s})\mu^{\boldsymbol{\tau}}|\mathcal{F}_t]$ denotes the restriction of the measure $Y_T(\boldsymbol{s})\mu^{\boldsymbol{\tau}}$ on $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$. Note that one has $E[Y_T(\boldsymbol{s})\mu^{\boldsymbol{\tau}}|\mathcal{F}_t] = E[Y_T(\boldsymbol{s})\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t]$.

We shall use the Radon-Nikodym derivative of random measures to interpret diverse conditional expectations.

Proposition 2.6 *Let $T \geq t \geq 0$. For any positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function $Y_T(\cdot)$ on $\Omega \times \mathbb{R}_+^n$, one has*

$$\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{F}_t] = \frac{\int_{\mathbb{R}_+^n} \mathbb{E}[Y_T(\boldsymbol{s})\mu^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s})}{\int_{\mathbb{R}_+^n} \mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})} \tag{7}$$

where the Radon-Nikodym derivative is taken on $\mathcal{F}_t$.

Remark 2.7 Note that $\int_{\mathbb{R}_+^n} \mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}) = \mathbb{P}(d\omega)$, the above result can be written as

$$\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{F}_t]\mathbb{P}(d\omega) = \int_{\mathbb{R}_+^n} \mathbb{E}[Y_T(s_I)\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s}).$$

Proof. Let h_t be a positive $\mathcal{F}_t$-measurable random variable, then

$$\int h_t(\omega)\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{F}_t]\mathbb{P}(d\omega) = \mathbb{E}\big[h_t\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{F}_t]\big] = \mathbb{E}[h_t Y_T(\boldsymbol{\tau})] = \int h_t(\omega)Y_T(\boldsymbol{s})\mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}).$$

Hence the equality (7) holds. □

Remark 2.8 In particular, the conditional expectation $\mathbb{E}[Y_T|\mathcal{F}_t]$ where Y_T is a positive $\mathcal{G}_T$-measurable random variable can be written in a decomposed form. In fact, by lemma 2.1, one has

$$Y_T = \sum_{I \subset \Theta} 1\!\!1_{A_T^I} Y_T^I(\tau_I) = \sum_{I \subset \Theta} 1\!\!1_{[0,T]^I \times]T,\infty[^{I^c}}(\boldsymbol{\tau}) Y_T^I(\tau_I),$$

where $Y_T^I(\cdot)$ is positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable. Hence Proposition 2.6 gives

$$\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{F}_t]\mathbb{P}(d\omega) = \sum_{I \subset \Theta} \int_{[0,T]^I \times]T,\infty[^{I^c}} \mathbb{E}[Y_T(s_I)\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s}).$$

3 Pricing with multiple defaults

For the purpose of pricing, let us consider a contingent claim sensitive to multiple defaults with the payoff function $Y_T(\boldsymbol{\tau})$ where $Y_T(\boldsymbol{s})$ is a positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function on $\Omega \times \mathbb{R}_+^n$, T being the maturity. Since $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ represents a family of default times, $Y_T(\boldsymbol{\tau})$ can describe a large class of financial products such as a basket

credit derivative, or a single-name contingent claim subjected to the default risks of multiple counterparties, or a basket European option with contagion risks etc. The price of this product is computed as the expectation $\mathbb{E}[Y_T(\boldsymbol{\tau})]$ under some risk-neutral probability measure. The dynamic price process given all market information at time $t \leq T$ is the conditional expectation $\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{G}_t]$. In this section, we shall present the evaluation formulas using the random measures.

3.1 General pricing formula

We suppose in this section that the (conditional) expectations are taken under some risk-neutral probability.

Theorem 3.1 *Let $T \geq t \geq 0$. For any positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function $Y_T(\cdot)$ on $\Omega \times \mathbb{R}_+^n$, the measure $\mu_t^{Y_T,I}$ is absolutely continuous with respect to μ_t^I. Moreover, the following equality holds*

$$\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{G}_t] = \sum_{I \subset \Theta} 1\!\!1_{A_t^I} \frac{d\mu_t^{Y_T,I}}{d\mu_t^I}(\omega, \tau_I). \tag{8}$$

Using the notations (4) and (6), the above equality can also be written as

$$\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{G}_t] = \sum_{I \subset \Theta} 1\!\!1_{A_t^I} \frac{\int_{]t,\infty[^{I^c}} \mathbb{E}[Y_T^I(s_I)\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^{I^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})}\Bigg|_{s_I=\tau_I} \tag{9}$$

where the Radon-Nikodym derivative is taken on the σ-algebra $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I)$.

Proof. By definition (3), one has that, for any $M \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I)$, $\mu_t^I(M) = 0$ if and only if $1\!\!1_M(\omega, s_I)1\!\!1_{]t,+\infty[^{I^c}}(s_{I^c}) = 0$, $\mu^{\boldsymbol{\tau}}$-a.e. Hence this implies $\mu_t^{Y_T,I}(M) = 0$.
On the set A_t^I, any $\mathcal{G}_t$-measurable test random variable can be written in the form $Z_t(\tau_I)$, where $Z_t(s_I)$ is positive and $\mathcal{F}_t \otimes \mathcal{B}(R_+^I)$-measurable. To prove (8), it suffices to establish

$$\mathbb{E}[1\!\!1_{A_t^I} Z_t(\tau_I) Y_T(\tau)] = \mathbb{E}\Big[1\!\!1_{A_t^I} Z_t(\tau_I) \frac{d\mu_t^{Y_T,I}}{d\mu_t^I}(\omega, \tau_I)\Big]. \tag{10}$$

One has

$$\begin{aligned}
&\mathbb{E}\Big[1\!\!1_{A_t^I} Z_t(\tau_I) \frac{d\mu_t^{Y_T,I}}{d\mu_t^I}(\omega, \tau_I)\Big] \\
&= \int 1\!\!1_{[0,t]^I}(s_I) 1\!\!1_{]t,+\infty[^{I^c}}(s_{I^c}) Z_t(s_I) \frac{d\mu_t^{Y_T,I}}{d\mu_t^I}(\omega, s_I)\mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}) \\
&= \int 1\!\!1_{[0,t]^I}(s_I) Z_t(s_I) \frac{d\mu_t^{Y_T,I}}{d\mu_t^I}(\omega, s_I)\mu_t^I(d\omega, ds_I) \\
&= \int 1\!\!1_{[0,t]^I}(s_I) Z_t(s_I) \mu_t^{Y_T,I}(d\omega, ds_I) \\
&= \int 1\!\!1_{[0,t]^I}(s_I) 1\!\!1_{]t,+\infty[^{I^c}}(s_{I^c}) Z_t(s_I) Y_T(\boldsymbol{s})\mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}),
\end{aligned}$$

where the first equality comes from the definition of $\mu^{\boldsymbol{\tau}}$, the second one comes from (3), the third one results from the definition of Radon-Nikodym derivative, and the last one comes from (5). Again by the definition of $\mu^{\boldsymbol{\tau}}$, the last formula equals the left side of (10). □

Remark 3.2 The following form of Theorem 3.1 will be useful. Let Y_T be a positive $\mathcal{G}_T$-measurable random variable, which is written as

$$Y_T = \sum_{I \subset \Theta} 1\!\!1_{A_T^I} Y_T^I(\tau_I),$$

where $Y_T^I(\cdot)$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable. Theorem 3.1 implies that

$$\begin{aligned}
\mathbb{E}[Y_T|\mathcal{G}_t] &= \sum_{I\subset\Theta} \mathbb{E}[1\!\!1_{A_T^I} Y_T^I(\tau_I)|\mathcal{G}_t] \\
&= \sum_{I\subset\Theta}\sum_{J\subset\Theta} 1\!\!1_{A_t^J} \frac{\int_{]t,\infty[^{J^c}} \mathbb{E}[1\!\!1_{[0,T]^I\times]T,\infty[^{I^c}}(\boldsymbol{s}) Y_T^I(s_I)\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^{J^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})} \\
&= \sum_{I\subset\Theta}\sum_{J\subset I} 1\!\!1_{A_t^J} \frac{\int_{]t,\infty[^{J^c}} 1\!\!1_{[0,T]^I\times]T,\infty[^{I^c}}(\boldsymbol{s}) \mathbb{E}[Y_T^I(s_I)\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^{J^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})} \\
&= \sum_{J\subset\Theta} 1\!\!1_{A_t^J} \sum_{I\supset J} \frac{\int_{]T,\infty[^{I^c}\times]t,T]^{I\setminus J}} \mathbb{E}[Y_T^I(s_I)\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^{J^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})},
\end{aligned} \tag{11}$$

where the last equality comes from an interchange of summation.

Inspired by [5], we consider the case where the density hypothesis holds. Let $\nu^{\boldsymbol{\tau}}$ be the marginal measure of $\mu^{\boldsymbol{\tau}}$ on $\mathcal{B}(\mathbb{R}_+^n)$, that is,

$$\nu^{\boldsymbol{\tau}}(U) = \mu^{\boldsymbol{\tau}}(\Omega \times U), \quad \forall\, U \in \mathcal{B}(\mathbb{R}_+^n).$$

Note that $\nu^{\boldsymbol{\tau}}$ is actually the law of $\boldsymbol{\tau}$.

Assumption 3.3 We say that $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ satisfies the *density hypothesis* if the measure $\mu^{\boldsymbol{\tau}}$ is absolutely continuous with respect to $\mathbb{P} \otimes \nu^{\boldsymbol{\tau}}$. We denote by $\alpha_t(\cdot)$ the density of $\mu^{\boldsymbol{\tau}}$ with respect to $\mathbb{P} \otimes \nu^{\boldsymbol{\tau}}$ on $(\Omega \times \mathbb{R}_+^n, \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n))$, where $t \in \mathbb{R}_+$.

Under the above density hypothesis, one has, for any positive Borel function on $\mathbb{R}_+^n$,

$$\mathbb{E}[f(\boldsymbol{\tau})|\mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\boldsymbol{s})\alpha_t(\boldsymbol{s})\nu^{\tau}(ds).$$

This relationship can also be written as

$$\mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}) = \alpha_t(\boldsymbol{s})\mathbb{P}(d\omega) \otimes \nu^{\boldsymbol{\tau}}(d\boldsymbol{s}). \tag{12}$$

Corollary 3.4 *We keep the notation of Theorem 3.1 and assume in addition the density hypothesis 3.3. Then*

$$\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{G}_t] = \sum_I \mathbb{1}_{A_t^I} \frac{\int_{]t,+\infty[^{I^c}} \mathbb{E}[Y_T(\boldsymbol{s})\alpha_T(\boldsymbol{s})|\mathcal{F}_t]\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}{\int_{]t,\infty[^{I^c}} \alpha_t(\boldsymbol{s})\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}\Big|_{s_I=\tau_I} \tag{13}$$

Proof. By the density hypothesis (12) and by (4), for any $I \subset \Theta$, one has

$$\mu_t^I(d\omega, ds_I) = \int_{]t,\infty[^{I^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}) = \int_{]t,\infty[^{I^c}} \alpha_t(\boldsymbol{s})\mathbb{P}(d\omega) \otimes \nu^{\boldsymbol{\tau}}(d\boldsymbol{s}).$$

Similarly,

$$\mu_t^{Y_T,I}(d\omega, ds_I) = \int_{]t,\infty[^{I^c}} \mathbb{E}[Y_T(s_I)\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s}) = \int_{]t,\infty[^{I^c}} \mathbb{E}[Y_T(s_I)\alpha_T(\boldsymbol{s})|\mathcal{F}_t]\mathbb{P}(d\omega)\otimes\nu^{\boldsymbol{\tau}}(d\boldsymbol{s}).$$

So

$$\frac{d\mu_t^{Y_T,I}}{d\mu_t^I} = \frac{\int_{]t,\infty[^{I^c}} \mathbb{E}[Y_T(\boldsymbol{s})\alpha_T(\boldsymbol{s})|\mathcal{F}_t]\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}{\int_{]t,\infty[^{I^c}} \alpha_t(\boldsymbol{s})\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}.$$

In fact, for any positive $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable function $h_t(\cdot)$, one has

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}_+^I} h_t(s_I)\frac{\int_{]t,+\infty[^{I^c}} \mathbb{E}[Y_T(\boldsymbol{s})\alpha_T(\boldsymbol{s})|\mathcal{F}_t]\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}{\int_{]t,\infty[^{I^c}} \alpha_t(\boldsymbol{s})\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}\mu_t^I(d\omega, ds_I)\\
=&\int_{\Omega\times\mathbb{R}_+^I\times]t,\infty[^{I^c}} h_t(s_I)\frac{\int_{]t,+\infty[^{I^c}} \mathbb{E}[Y_T(\boldsymbol{s})\alpha_T(\boldsymbol{s})|\mathcal{F}_t]\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}{\int_{]t,\infty[^{I^c}} \alpha_t(\boldsymbol{s})\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}\alpha_t(\boldsymbol{s})\mathbb{P}(d\omega)\otimes\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})\\
=&\int_\Omega \mathbb{P}(d\omega)\int_{\mathbb{R}_+^I} h_t(s_I)\frac{\int_{]t,+\infty[^{I^c}} \mathbb{E}[Y_T(\boldsymbol{s})\alpha_T(\boldsymbol{s})|\mathcal{F}_t]\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}{\int_{]t,\infty[^{I^c}} \alpha_t(\boldsymbol{s})\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}\int_{]t,\infty[^{I^c}} \alpha_t(\boldsymbol{s})\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})\\
=&\int_\Omega \mathbb{P}(d\omega)\int_{\mathbb{R}^I\times]t,\infty[^{I^c}} h_t(s_I)\mathbb{E}[Y_T(\boldsymbol{s})\alpha_T(\boldsymbol{s})|\mathcal{F}_t]\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})\\
=&\int_{\Omega\times\mathbb{R}_+^I\times]t,\infty[^{I^c}} h_t(s_I)Y_T(\boldsymbol{s})\alpha_T(\boldsymbol{s})\mathbb{P}(d\omega)\otimes\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})\\
=&\int_{\Omega\times\mathbb{R}_+^I} h_t(s_I)\mu_t^{Y_T,I}(d\omega, ds_I),
\end{aligned}$$

where the first and the last equality come from the density hypothesis, the second and the fourth ones come from Fubini's theorem. Thus the equality (13) follows from Theorem 3.1. □

3.2 Pricing of credit portfolio derivatives

We now apply the previous pricing formulas to the two important types of credit portfolio derivatives: the k^{th}-to-default swaps and the CDOs.

3.2.1 Basket default swaps

A k^{th}-to-default swap provides to its buyer the protection against the k^{th} default of the underlying portfolio. Let $(\tau_{(k)})_{k\in\Theta}$ be the ordered set of $\boldsymbol{\tau} = (\tau_i)_{i\in\Theta}$, that is, $\tau_{(1)} < \cdots < \tau_{(n)}$. The protection buyer pays a regular premium until the occurrence of the k^{th} default time $\tau_{(k)}$ or until the maturity T if there are less than k defaults before T. In return, the protection seller pays the loss $1 - R_{(k)}$ where $R_{(k)}$ is the recovery rate if $\tau_{(k)} \leq T$, and pays zero otherwise. So the key term for evaluating such a product is the indicator default process $1\!\!1_{\{\tau_{(k)}\leq T\}}$ with respect to the market filtration $\mathcal{G}_t$.

Proposition 3.5 *For any* $t \leq T$,

$$\mathbb{E}[1\!\!1_{\{\tau_{(k)}>T\}}|\mathcal{G}_t] = \sum_{|J|<k} 1\!\!1_{A_t^J} \sum_{I\supset J,\,|I|<k} \frac{\int_{]T,\infty[^{I^c}} \int_{]t,T]^{I\setminus J}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^{J^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})}\Bigg|_{s_J=\tau_J}. \tag{14}$$

Proof. Observe that $1\!\!1_{\{\tau_{(k)}>T\}} = \sum_{|I|<k} 1\!\!1_{A_T^I}$. By Theorem 3.1, one obtains

$$\mathbb{E}[1\!\!1_{A_T^I}|\mathcal{G}_t] = \sum_{J\subset I} 1\!\!1_{A_t^J} \frac{\int_{]T,\infty[^{I^c}} \int_{]t,T]^{I\setminus J}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^{J^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})}\Bigg|_{s_J=\tau_J}.$$

By taking the sum over I such that $|I| \leq k$ and by interchanging the summations, one gets (14). $\square$

Remark 3.6 Among the basket default swaps, the *first-to-default swap* is the most important one. In this case, $k = 1$. Proposition 3.5 leads to

$$\mathbb{E}[1\!\!1_{\{\tau_{(1)}>T\}}|\mathcal{G}_t] = 1\!\!1_{\{\tau_{(1)}>t\}} \frac{\int_{]T,\infty[^n} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^n} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})} = 1\!\!1_{\{\tau_{(1)}>t\}} \frac{\mathbb{E}[1\!\!1_{\{\tau_{(1)}>T\}}|\mathcal{F}_t]}{\mathbb{E}[1\!\!1_{\{\tau_{(1)}>t\}}|\mathcal{F}_t]}.$$

The last equality is a well-known result (e.g. [1]).

3.2.2 CDO tranches

A CDO is a structured credit derivative based on a large pool of underlying assets and containing several tranches. For the pricing of a CDO tranche, the term of interest is the cumulative loss of the portfolio $l_t = \sum_{i=1}^n R_i 1\!\!1_{\tau_i\leq t}$, R_i being the recovery rate of τ_i. A tranche of CDO is specified by an interval corresponding to the partial loss of the portfolio. The two threshold values, a upper value a and a lower one b, defines a tranche of CDO and the loss on the tranche is given as a call spread written on the loss process l_t with strike values a and b. Therefore to obtain the dynamics of the CDO prices, we shall consider $\mathbb{E}[(l_T - a)^+|\mathcal{G}_t]$.

On the market, it is a standard hypothesis to suppose that the recovery rate for each underlying name is constant (equal to 40% in practice). We make this hypothesis below

and discuss further on in Remark 4.10 the case where the recovery rates R_i are random variables.

The following result allows us to deduce the CDO prices using the k^{th}-to default swaps.

Proposition 3.7 *Assume that $R_i = R$ is constant for all $i \in \Theta$, then*

$$(l_T - a)_+ = R \sum_{k \geq a/R} \min(k - \frac{a}{R}, 1)\, \mathbb{1}_{\{\tau_{(k)} \leq T\}}. \tag{15}$$

Moreover,

$$\mathbb{E}[(l_T - a)_+ | \mathcal{G}_t] = R \sum_{k \geq a/R} \min(k - \frac{a}{R}, 1) \Big[1 - \sum_{|J|<k} \mathbb{1}_{A_t^J} \sum_{I \supset J, |I|<k} \frac{\int_{]T,\infty[^{I^c}} \int_{]t,T]^{I \setminus J}} \mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^{J^c}} \mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})} \Big|_{s_J = \tau_J} \Big].$$

Proof. Notice that for any $m \in \Theta$, the following equality holds

$$(m - \frac{a}{R})_+ = \sum_{k \in \Theta, \frac{a}{R} \leq k \leq m} \min(k - \frac{a}{R}, 1).$$

Hence

$$(\sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq T\}} - \frac{a}{R})_+ = \sum_{k \in \Theta, \frac{a}{R} \leq k} \min(k - \frac{a}{R}, 1) \mathbb{1}_{\{k \leq \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq T\}}\}}.$$

Since $\{k \leq \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq T\}}\} = \{\tau_{(k)} \leq T\}$, we obtain (15) and

$$\mathbb{E}[(l_T - a)_+ | \mathcal{G}_t] = \sum_{k \geq \frac{a}{R}} \min(k - \frac{a}{R}, 1)\, \mathbb{E}[\mathbb{1}_{\{\tau_{(k)} \leq T\}} | \mathcal{G}_t].$$

By Proposition 3.5, the result holds. □

4 The contagion risk model

In this section, we are interested in the contagion risks with multiple defaults. One observation during the financial crisis is that one default event may have impact on other remaining firms and often causes important losses on the asset values of its counterparties. We shall propose a contagion model to take into consideration this phenomenon.

4.1 Preliminaries

We begin by generalizing Lemmas 2.1 and 2.2 to the case of processes. Denote by $\mathcal{O}_{\mathbb{F}}$ (resp. $\mathcal{P}_{\mathbb{F}}$) the optional (resp. predictable) σ-field on $\Omega \times \mathbb{R}_+$.

Lemma 4.1 *1) Any $\mathbb{G}$-optional process Y can be written as $Y_t = \sum_I 1\!\!1_{A_t^I} Y_t^I(\tau_I)$ where $Y^I(\cdot)$ is an $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+^I$.*
2) Any $\mathbb{G}$-predictable process Y can be written as $Y_t = \sum_I 1\!\!1_{A_{t-}^I} Y_t^I(\tau_I)$ where $Y^I(\cdot)$ is a $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+^I$.

Proof. 1) It suffices to consider $Y = Z 1\!\!1_{[\![s,\infty[\![}$, Z being a $\mathcal{G}_s$-measurable random variable. By Lemma 2.1, for any $I \subset \Theta$, there exists an $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable function $Z^I(\cdot)$ such that

$$Z = \sum_{I \subset \Theta} 1\!\!1_{A_s^I} Z^I(\tau_I).$$

We define

$$Y_t^I(s_I) := \begin{cases} \sum_{J \subset I} Z^J(s_J) 1\!\!1_{[0,s]^J}(s_J) \prod_{i \in I \setminus J} 1_{\{s < s_i \leq t\}} & \text{if } t \geq s, \\ 0 & \text{if } t < s. \end{cases}$$

Notice that the process $Y^I(s_I)$ is right continuous for any s_I. Hence Y^I is $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable. By the equality

$$\sum_{I \supset J} 1\!\!1_{A_t^I} \Big(\prod_{i \in I \setminus J} 1\!\!1_{\{s < \tau_i \leq t\}} \Big) 1\!\!1_{[0,s]^J}(\tau_J) = 1\!\!1_{A_s^J}$$

which holds for any $J \subset \Theta$ and any $t \in [s, +\infty[$, one can verify that $Y_t = \sum_I 1\!\!1_{A_t^I} Y^I(\tau_I)$.
2) By using Lemma 2.2, a variant of the above argument leads to the predictable version of 1).

□

4.2 The model setup

We consider a portfolio of N assets, whose value process is denoted by S which is an N-dimensional $\mathbb{G}$-adapted process. The process S has the following decomposed form

$$S_t = \sum_{I \subset \Theta} 1\!\!1_{A_t^I} S_t^I(\tau_I),$$

where $S_t^I(s_I)$ is $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable and takes values in $\mathbb{R}_+^N$, representing the asset values given the past default events $\tau_I = s_I$. Note that S_t depends on the value of $S_t^I(\cdot)$ only on the set A_t^I, that is, only when $t \geq \max s_I$. Hence we may assume in convention that $S_t^I(s_I) = 0$ for $t < s_{\vee I}$ where $s_{\vee I} := \max s_I$ with $s_\emptyset = 0$.

We suppose that the dynamics of S^I is given by

$$dS_t^I(s_I) = S_t^I(s_I) * (\mu_t^I(s_I) dt + \Sigma_t^I(s_I) dW_t), \quad t > s_{\vee I} \tag{16}$$

where W is a N-dimensional Brownian motion with respect to the filtration $\mathbb{F}$, the coefficients $\mu_t^I(s_I)$ and $\Sigma_t^I(s_I)$ are $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable and bounded. Note that for two vectors $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N)$ in $\mathbb{R}^N$, the expression $x * y$ denotes the vector $(x_1 y_1, \cdots, x_N y_N)$.

We also suppose that one default event induces a jump on each remaining asset in the portfolio. More precisely, for any $I \neq \emptyset$, let

$$S_{s_{\vee I}}^I(s_I) = S_{s_{\vee I}-}^J(s_J) * (\mathbf{1} - \gamma_{s_{\vee I}}^{J,k}(s_J)), \tag{17}$$

where $k = \min\{i \in I | s_i = s_{\vee I}\}$, $J = I \setminus \{k\}$ and $\gamma^{J,k}(\cdot)$ is $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^J)$-measurable, representing the jump at $\tau_{\vee I} = s_{\vee I}$ given the past default events $\tau_I = s_I$ with the last arriving default τ_k. The specification of k and J is for treating the case where several components in s_I are equal to $s_{\vee I}$. We remind that this convention is harmless since we have assumed that $\tau_i \neq \tau_j$ a.s. if $i \neq j$.

The value of $S_T^I(\cdot)$ is determined by $S_{s_{\vee I}}^I(\cdot)$ and the coefficients $\mu^I(\cdot)$ and $\Sigma^I(\cdot)$. From a recursive point of view, $S_T^I(\cdot)$ actually depends on the initial value S_0 and the coefficients indexed by all $J \subset I$, together with the jumps at defaults.

Remark 4.2 We do not suppose that $\mathbb{F}$ is generated by the Brownian motion W, allowing for some further generalizations. Furthermore, we do not specify the set of assets and the set of defaultable names, which permits to include a large family of models.

We give below several examples.

Example 4.3 (Exogenous portfolio) We consider an exogenous investment portfolio, that is, the underlying assets in the investment portfolio are not included in the defaultable portfolio. In this case, the default family contains often highly risky names while the investors prefer to choose assets in relatively better situations. However, these assets are influenced by the defaults. This is the case considered in [11].

Example 4.4 (Multilateral counterparty risks) The defaults family and the assets family coincide, each underlying name subjected to the default risk of itself and to the counterparty default risks of the other names of the portfolio. For each name $i \in \Theta$, denote by S^i its value process and by τ_i its default time. We suppose that the value of S^i drops to zero at the default time τ_i, and at the default times τ_j where $j \neq i, j \in \Theta$, the value of S^i has a jump. So S^i has the decomposed form

$$S_t^i = \sum_{I \not\ni i} 1\!\!1_{A_t^I} S_t^{i,I}(\tau_I),$$

where $S_t^{i,I}(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable. For $I \subset \Theta$ such that $i \notin I$, let the dynamics of $S^{i,I}$ satisfy

$$dS_t^{i,I}(s_I) = S_t^{i,I}(s_I)\big(\mu_t^{i,I}(s_I)dt + \sigma_t^{i,I}(s_I)dW_t^i\big), \quad t > s_{\vee I}$$

where $(W^1, \cdots, W^n)$ is an n-dimensional Brownian motion with covariance matrice Σ, $\mu_t^{i,I}(\cdot)$ and $\sigma_t^{i,I}(\cdot)$ are $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(R_+^I)$-measurable and bounded. In addition, we suppose that for any $I \neq \emptyset$,

$$S_{s_{\vee I}}^{i,I}(s_I) = S_{s_{\vee I}-}^{i,J}(s_J)(1 - \gamma_{s_{\vee I}}^{i,J,k}(s_J))$$

where $k = \min\{i \in I | s_i = s_{\vee I}\}$, $J = I \setminus \{k\}$.

4.3 A recursive optimization methodology

In this subsection, we consider the optimal investment problem in the contagion risk model. Following the recursive point of view on the successive defaults in [5], we have proposed in [11] a two-steps — before-default and after-default — optimization procedure. By using the density approach introduced in [4], the optimizations are effectuated with respect to the default-free filtration $\mathbb{F}$ instead of the global one $\mathbb{G}$. This methodology has been adopted recently in [13] to add a random mark at each default time by using the joint density of the ordered default times and the marks. We discuss this case as an application of constrained optimization problems (Remark 4.10) using auxiliary filtrations.

We apply now this recursive optimization methodology to the contagion model described previously. We shall assume the density hypothesis 3.3 in the sequel with $\nu^{\boldsymbol{\tau}}$, the law of $\boldsymbol{\tau}$, being the Lebesgue measure. Namely, we assume that the random measure of $\boldsymbol{\tau}$ is absolutely continuous with respect to $\mathbb{P} \otimes d\boldsymbol{s}$ on σ-field $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$, $t \in \mathbb{R}_+$, where $d\boldsymbol{s}$ is the probability law of $\boldsymbol{\tau}$. Denote by $\alpha_t(\boldsymbol{s})$ the density of $\mu^{\boldsymbol{\tau}}$ on $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$. By taking a suitable version, one has that $\alpha(\boldsymbol{s})$ is an $\mathbb{F}$-martingale for any $\boldsymbol{s} \in \mathbb{R}_+^n$. Note that we do not suppose that the defaults are ordered.

Let us consider an investor who holds a portfolio of assets, each one subjected to contagion risks. The value process of the assets is supposed to satisfy (16) and (17). The wealth process of this investor is described by a positive $\mathbb{G}$-adapted process X and the allocation of the portfolio is chosen by the criterion of maximizing the utility of the terminal wealth at a finite horizon T. So we are interested in the optimization problem $\mathbb{E}[U(X_T)]$ where U is a utility function satisfying the conditions: strictly increasing, strictly concave, C^1 on $\mathbb{R}_+$ and the Inada conditions $U'(0_+) = \infty$, $U'(\infty) = 0$.

The portfolio of assets is characterized by a $\mathbb{G}$-predictable process π, representing the proportion of the wealth invested on each asset. By Lemma 4.1, the process π has the decomposed form $\pi_t = \sum_I \mathbb{1}_{A_{t-}^I} \pi_t^I(\tau_I)$ where $\pi^I(\cdot)$ is $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable, representing the investment strategy given the defaults τ_I. Hence, to determine an investment strategy π is equivalent to find a family $(\pi^I(\cdot))_{I \subset \Theta}$.

Similarly, the wealth process X has the decomposed form $X_t = \sum \mathbb{1}_{A_t^I} X_t^I(\tau_I)$ where $X^I(\cdot)$ is $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable, representing the wealth given τ_I. In view of (16), the wealth process dynamics is given by

$$dX_t^I(s_I) = X_t^I(s_I)\pi_t^I(s_I) \cdot \big(\mu_t^I(s_I)dt + \Sigma_t^I(s_I)dW_t\big), \quad t > s_{\vee I}. \tag{18}$$

Note that for two vectors $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N)$ in $\mathbb{R}^N$, the expression $x \cdot y$ denotes $x_1 y_1 + \cdots + x_N y_N$. By (17), we have for any $I \neq \emptyset$,

$$X^I_{s_{\vee I}}(s_I) = X^J_{s_{\vee I}-}(s_J)(1 - \pi^J_{s_{\vee I}}(s_J) \cdot \gamma^{J,k}_{s_{\vee I}}), \tag{19}$$

where $k = \min\{i \in I | s_i = s_{\vee I}\}$ and $J = I \setminus \{k\}$. To ensure that the wealth is positive, we need to suppose that the cumulative (proportional) losses caused by one default on all remaining names is smaller than 1.

This following simple result is useful for the recursive optimization.

Lemma 4.5 *For any $T \in \mathbb{R}_+$,*

$$\mathbb{E}[U(X_T)] = \sum_{I \subset \Theta} \int_{[0,T]^I \times]T,\infty[^{I^c}} \mathbb{E}[U(X^I_T(s_I))\alpha_T(\boldsymbol{s})]d\boldsymbol{s}. \tag{20}$$

Proof. We use the decomposed form of X_T in Remark 2.8 and take iterated conditional expectation to obtain

$$\mathbb{E}[U(X_T)] = \sum_{I \subset \Theta} \mathbb{E}[1\!\!1_{A^I_T} U(X^I_T(\tau_I))] = \sum_{I \subset \Theta} \mathbb{E}\big[\mathbb{E}[1\!\!1_{A^I_T} U(X^I_T(\tau_I)) | \mathcal{F}_T]\big].$$

The lemma then follows by definition of the density. □

We introduce the admissible strategy sets.

Definition 4.6 For $I \subset \Theta$ and $s_I \in [0,T]^I$, let $\mathcal{A}^I(s_I)$ be the set of $\mathbb{F}$-predictable processes $\pi^I(s_I)$ such that the following two conditions are satisfied:

1) $\int_0^T |\pi^I_t(s_I)\sigma^I_t(s_I)|^2 dt < \infty$;

2) in the case where $I \neq \Theta$, for any $i \notin I$ and any $s_i \in]s_{\vee I}, T]$, one has $\pi^I_{s_i}(s_I) \cdot \gamma^{I,i}_{s_i} < 1$.

Denote by $\mathcal{A} = \{(\pi^I(\cdot))_{I \subset \Theta}\}$ the set of strategy families $\pi = (\pi^I(\cdot))_{I \subset \Theta}$, where $\pi^I(\cdot)$ is a $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^I_+)$-measurable function such that for any $s_I \in [0,T]^I$, the process $\pi^I(s_I)$ is in $\mathcal{A}^I(s_I)$. We say that π is admissible if $\pi = (\pi^I(\cdot))_{I \subset \Theta} \in \mathcal{A}$.

For our recursive methodology, it will also be useful to consider all the strategies after the defaults $\tau_I = s_I$. We introduce the corresponding admissible sets below.

Definition 4.7 For any $I \subset \Theta$ and any $s_I \in [0,T]^I$, let $\mathcal{A}^{\supset I}(s_I)$ be the set of families $(\pi^K(s_I, \cdot))_{K \supset I}$, where $\pi^K(s_I, \cdot)$ is a $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^{K \setminus I}_+)$-measurable function such that $\pi^K(s_K) \in \mathcal{A}^K(s_K)$ for any $s_{K \setminus I} \in [s_{\vee I}, T]^{K \setminus I}$.

We define the set $\mathcal{A}^{\supsetneq I}(s_I)$ in a similar way. Note that any family $\pi^{\supset I}$ in $\mathcal{A}^{\supset I}(s_I)$ can be written as $(\pi^I, \pi^{\supsetneq I})$ with $\pi^{\supsetneq I} \in \mathcal{A}^{\supsetneq I}(s_I)$ and $\pi^I \in \mathcal{A}^I(s_I)$.

Let us now consider the maximization of the utility function on the terminal value of the wealth process

$$J(x,\pi) := \mathbb{E}[U(X_T)]_{X_0=x}. \tag{21}$$

We shall treat the optimization problem in a backward and recursive way. To this end, we introduce some notations. Let

$$J_\Theta(x,\boldsymbol{s},\pi^\Theta) := \mathbb{E}[U(X_T^\Theta(\boldsymbol{s}))\alpha_T(\boldsymbol{s})\,|\,\mathcal{F}_{s_{\vee\Theta}}]_{X^\Theta_{s_{\vee\Theta}}(\boldsymbol{s})=x},$$

and

$$V_\Theta(x,\boldsymbol{s}) = \operatorname*{esssup}_{\pi^\Theta\in\mathcal{A}^\Theta(\boldsymbol{s})} J_\Theta(x,\boldsymbol{s},\pi^\Theta).$$

We define recursively for $I\subset\Theta$,

$$\begin{aligned} J_I(x,s_I,\pi^I) := \mathbb{E}\Big[&U(X_T^I(s_I))\int_{]T,+\infty[^{I^c}}\alpha_T(\boldsymbol{s})ds_{I^c}\\ &+\sum_{i\in I^c}\int_{]s_{\vee I},T]}V_{I\cup\{i\}}\big(X_{s_i}^{I\cup\{i\}}(s_{I\cup\{i\}}),s_{I\cup\{i\}}\big)ds_i\,\Big|\,\mathcal{F}_{s_{\vee I}}\Big]_{X^I_{s_{\vee I}}(s_I)=x}, \end{aligned} \tag{22}$$

and correspondingly

$$V_I(x,s_I) := \operatorname*{esssup}_{\pi^I\in\mathcal{A}^I(s_I)} J_I(x,s_I,\pi^I). \tag{23}$$

Remark 4.8 Note that viewed from the initial time $t=0$, the value of $X_T^I(s_I)$ depends on X_0 and all strategies π^J, $J\subset I$. However, viewed from the last arriving default $\tau_{\vee I}$, its value depends only on the strategy π^I if the value of $X^I_{s_{\vee I}}(s_I)$ is given.

The above constructions provide us a family of optimization problems $(V_I(x,s_I))_{I\subset\Theta}$. Notice that at each step, the optimization problem V_I involves the resolution of other ones $V_{I\cup\{i\}}$. The whole system need to be dealt with in a recursive manner backwardly, each problem concerning the filtration $\mathbb{F}$ and the time interval $[s_{\vee I},T]$. By resolving recursively the problems, we can obtain a family of optimal strategies $(\widehat{\pi}^I(\cdot))_{I\subset\Theta}$. The following theorem shows that the global optimization problem, which consists of finding the optimal strategy $\widehat{\pi}\in\mathcal{A}$ for (21) is equivalent to finding $(\widehat{\pi}_I(\cdot))_{I\subset\Theta}$.

With the above notations, one has in particular

$$J_\emptyset(x,\pi^\emptyset) = \mathbb{E}\Big[U(X_T^\emptyset)\int_{]T,\infty[^n}\alpha_T(\boldsymbol{s})d\boldsymbol{s}+\sum_{i=1}^n\int_{]0,T]}V_{\{i\}}(X_{s_i}^{\{i\}}(s_i),s_i)ds_i\Big]_{X_0=x}$$

and $V_\emptyset(x) = \sup_{\pi^\emptyset\in\mathcal{A}^\emptyset} J_\emptyset(x,\pi^\emptyset)$.

Theorem 4.9 *Suppose that $V^I(x, s_I) < \infty$ a.s. for any $I \subset \Theta$, any $x > 0$ and $s_I \in [0, T]^I$, then*

$$\sup_{\pi \in \mathcal{A}} J(x, \pi) = V_\emptyset(x). \tag{24}$$

Proof. For any $I \subset \Theta$, we deduce from a backward point of view and introduce

$$\widetilde{J}_I(y, s_I, \pi^{\supset I}) = \mathbb{E}\Big[\sum_{K \supset I} \int_{]T,\infty[^{K^c}} \int_{]s_{\vee I}, T]^{K \setminus I}} U(X_T^K(s_K))\alpha_T(\boldsymbol{s}) ds_{I^c} \,\Big|\, \mathcal{F}_{s_{\vee I}}\Big], \tag{25}$$

where $X^I_{s_{\vee I}}(s_I) = y$ and $\pi^{\supset I}$ is an element in $\mathcal{A}^{\supset I}(s_I)$ (see Definition 4.7). Note that the value of $\widetilde{J}_I(y, s_I, \pi^{\supset I})$ depends on y, s_I and on the choice of $\pi^K(t_K)$ with $K \supset I$ and $t_I = s_I$. We shall prove by induction the equality

$$\operatorname*{esssup}_{\pi^{\supset I} \in \mathcal{A}^{\supset I}(s_I)} \widetilde{J}_I(y, s_I, \pi^{\supset I}) = V_I(y, s_I). \tag{26}$$

By Lemma 4.5, we have $\widetilde{J}_\emptyset(y, \pi) = \mathbb{E}[U(X_T)]_{X_0 = y}$. So the particular case of (26) when $I = \emptyset$ is just what need to be proved.

We proceed by induction on I and begin by $I = \Theta$. Observe that $\widetilde{J}_\Theta(y, \boldsymbol{s}, \pi^{\supset \Theta}) = J_\Theta(y, \boldsymbol{s}, \pi^\Theta)$. Hence the equality (26) holds true by definition when $I = \Theta$. Let I be a proper subset of Θ. Assume that we have proved (26) for all $K \supsetneq I$. We claim and show below that

$$\begin{aligned}\widetilde{J}_I(y, s_I, \pi^{\supset I}) &= \mathbb{E}\Big[U(X_T^I(s_I)) \int_{]T,\infty[^{I^c}} \alpha_T(\boldsymbol{s}) ds_{I^c} \\ &\quad + \sum_{i \in I^c} \int_{]s_{\vee I}, T]} \widetilde{J}_{I \cup \{i\}}(X_{s_i}^{I \cup \{i\}}(s_{I \cup \{i\}}), s_{I \cup \{i\}}, \pi^{\supset I \cup \{i\}}) ds_i | \mathcal{F}_{s_{\vee I}}\Big]_{X^I_{s_{\vee I}}(s_I) = y}.\end{aligned} \tag{27}$$

In fact, by (25), the second term in the right-hand side of (27) equals

$$\begin{aligned}&\sum_{i \in I^c} \int_{]s_{\vee I}, T]} ds_i \mathbb{E}[\widetilde{J}_{I \cup \{i\}}(X_{s_i}^{I \cup \{i\}}(s_{I \cup \{i\}}), s_{I \cup \{i\}}, \pi^{\supset I \cup \{i\}}) | \mathcal{F}_{s_{\vee I}}] \\ &= \sum_{i \in I^c} \sum_{K \supset I \cup \{i\}} \int_{]T,\infty[^{K^c}} \int_{]s_{\vee I}, T]^{\{i\}}} \int_{]s_i, T]^{K \setminus (I \cup \{i\})}} \mathbb{E}[U(X_T^K(s_K))\alpha_T(\boldsymbol{s}) | \mathcal{F}_{s_{\vee I}}] ds_{I^c} \\ &= \sum_{K \supsetneq I} \sum_{i \in K \setminus I} \int_{]T,\infty[^{K^c}} \int_{]s_{\vee I}, T]^{\{i\}}} \int_{]s_i, T]^{K \setminus (I \cup \{i\})}} \mathbb{E}[U(X_T^K(s_K))\alpha_T(\boldsymbol{s}) | \mathcal{F}_{s_{\vee I}}] ds_{I^c}.\end{aligned}$$

We consider, for any $i \in K \setminus I$, the set $]s_{\vee I}, T]^{\{i\}} \times]s_i, T]^{K \setminus (I \cup \{i\})}$. Note that the subsets of $]s_{\vee I}, T]^{K \setminus I}$ of the following form

$$\Gamma_i := \{s_{K \setminus I} \,|\, \forall\, j \in K \setminus (I \cup \{i\}),\ s_{\vee I} < s_i < s_j \leq T\}.$$

are disjoint, in addition, the set

$$]s_{\vee I}, T]^{K\setminus I} \setminus \bigcup_{i \in K\setminus I} \Gamma_i$$

is negligible for the Lebesgue measure. Hence we obtain

$$\begin{aligned}&\sum_{i\in I^c} \int_{]s_{\vee I},T]} ds_i \mathbb{E}[\widetilde{J}_{I\cup\{i\}}(X_{s_i}^{I\cup\{s\}}(s_{I\cup\{i\}}), s_{I\cup\{i\}}, \pi^{\supset I\cup\{i\}})|\mathcal{F}_{s_{\vee I}}]\\ &= \sum_{K \supsetneq I} \int_{]T,\infty[^{K^c}} \int_{]s_{\vee I},T]^{K\setminus I}} \mathbb{E}[U(X_T^K(s_K))\alpha_T(\boldsymbol{s})|\mathcal{F}_{s_{\vee I}}] ds_{I^c},\end{aligned}$$

and hence (27) is established.

By the induction hypothesis, one has

$$\operatorname*{esssup}_{\pi^{\supset I\cup\{i\}} \in \mathcal{A}^{\supset I\cup\{i\}}(s_{I\cup\{i\}})} \widetilde{J}_{I\cup\{i\}}(y, s_{I\cup\{i\}}, \pi^{\supset I\cup\{i\}}) = V_{I\cup\{i\}}(y, s_{I\cup\{i\}}). \tag{28}$$

Hence we have by (27)

$$\begin{aligned}\operatorname*{esssup}_{\pi^{\supset I} \in \mathcal{A}^{\supset I}(s_I)} \widetilde{J}_I(y, s_I, \pi^{\supset I}) &\le \int_{]T,\infty[^{I^c}} \mathbb{E}[U(X_T^I(s_I))\alpha_T(\boldsymbol{s})|\mathcal{F}_{s_{\vee I}}] ds_{I^c}\\ &+ \sum_{i\in I^c} \int_{]s_{\vee I},T]} \mathbb{E}[V_{I\cup\{i\}}(X_{s_i}^{I\cup\{i\}}(s_{I\cup\{i\}}), s_{I\cup\{i\}})|\mathcal{F}_{s_{\vee I}}] ds_i,\end{aligned}$$

which, together with the definitions (22) and (23), implies

$$\operatorname*{esssup}_{\pi^{\supset I} \in \mathcal{A}^{\supset I}(s_I)} \widetilde{J}_I(y, s_I, \pi^{\supset I}) \le V_I(y, s_I).$$

We still suppose the induction hypothesis (28) for the converse. For $i \notin I$, $s_{I\cup\{i\}} \in [0,T]^{I\cup\{i\}}$, $\varepsilon > 0$ and $z \in \mathbb{R}$, there exists a family $\pi_{\varepsilon,(z,i)}^{\supset I\cup\{i\}} \in \mathcal{A}^{\supset I\cup\{i\}}(s_{I\cup\{i\}})$ such that

$$\widetilde{J}_{I\cup\{i\}}\Big(z, s_{I\cup\{i\}}, \pi_{\varepsilon,(z,i)}^{\supset I\cup\{i\}}\Big) \ge V_{I\cup\{i\}}(z, s_{I\cup\{i\}}) - \varepsilon.$$

We fix s^I, $X_{s_{\vee I}}^I(s_I) = y$ and $\pi^I \in \mathcal{A}^I(s_I)$. By a measurable selection result, we can choose a family $\pi_\varepsilon^{\supsetneq I} \in \mathcal{A}^{\supsetneq I}(s_I)$ such that

$$\pi_\varepsilon^K(s_K) = \pi_{\varepsilon,(z_i,i)}^K(s_K),$$

where i, z_i and $s_{K\setminus I}$ satisfy the following conditions:

(1) $i \in I^c$, $s_{\vee I} < s_i$ and $s_i < s_j$ for all $j \in K \setminus (I \cup \{i\})$,

(2) $z_i = X_{s_i}^{I\cup\{i\}}(s_{I\cup\{i\}})$.

Note that the value of $\widetilde{J}_{I\cup\{i\}}(z_i, s_{I\cup\{i\}}, \pi^{\supset I\cup\{i\}})$ only depends on z_i, $s_{I\cup\{i\}}$ and the choice of $\pi^K(t_K)$ for those t_K such that $t_{I\cup\{i\}} = s_{I\cup\{i\}}$. We obtain that

$$\widetilde{J}_{I\cup\{i\}}(z_i, s_{I\cup\{i\}}, \pi_\varepsilon^{\supset I\cup\{i\}}) \geq V_{I\cup\{i\}}(z_i, s_{I\cup\{i\}}) - \varepsilon.$$

This implies, by comparing (22) and (27), that

$$\widetilde{J}_I(y, s_I, (\pi^I, \pi_\varepsilon^{\supsetneq I})) \geq J_I(y, s_I, \pi^I) - \varepsilon T|I^c|.$$

By taking the essential supremum over $\pi^I \in \mathcal{A}^I(s_I)$, one obtains

$$\operatorname*{esssup}_{\pi^{\supset I}\in\mathcal{A}^{\supset I}(s_I)} \widetilde{J}_I(y, s_I, \pi^{\supset I}) \geq V_I(y, s_I) - \varepsilon T|I^c|.$$

Since ε is arbitrary, we get the inequality

$$\operatorname*{esssup}_{\pi^{\supset I}\in\mathcal{A}^{\supset I}(s_I)} \widetilde{J}_I(y, s_I, \pi^{\supset I}) \geq V_I(y, s_I).$$

We hence established the equality (26). □

The existence and the explicit resolution of the optimization problems $(V_I(x, s_I))_{I\subset\Theta}$ will be discussed in detail in a companion paper.

Remark 4.10 It is often useful to consider strategies with constraints. In this case, the admissible set $\mathcal{A}^I_\circ(s_I)$ for $\pi^I(s_I)$, $s_I \in [0,T]^I$ is a subset of $\mathcal{A}^I(s_I)$ and the admissible trading strategy set $\mathcal{A}_\circ$ for π is defined similarly : $\mathcal{A}_\circ = \{(\pi^I(\cdot))_{I\subset\Theta}\}$ such that for any $s_I \in [0,T]^I$, $\pi^I(s_I) \in \mathcal{A}_\circ(s_I)$. We can also define $\mathcal{A}_\circ^{\supset I}(s_I)$ and $\mathcal{A}_\circ^{\supsetneq I}(s_I)$ in a similar way. Note that Theorem 4.9 still holds for the constrained strategy. More precisely, let us introduce in a backward and recursive way $J_I^\circ(x, s_I, \pi^I)$ similarly as in (22) and let $V_I^\circ(x, s_I) = \operatorname{esssup}_{\pi^I\in\mathcal{A}_I^\circ} J_I^\circ(x, s_I, \pi^I)$. Then

$$\sup_{\pi\in\mathcal{A}_\circ} J(x, \pi) = V_\emptyset^\circ(x).$$

As an application, we consider the case where the losses at default times $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ are associated to some $\mathcal{G}$-measurable random variables $\boldsymbol{L} = (L_1, \cdots, L_n)$. This is the case studied in [13] supposing $\boldsymbol{\tau}$ is a family of ordered default times and using the joint density of $(\boldsymbol{\tau}, \boldsymbol{L})$ with respect to the default-free filtration. We recall briefly this model and show that it can be considered as a constrained optimal problem mentioned in the above remark.

Example 4.11 Let $\mathbb{F}^\circ$ be a default-free filtration. The default information contains the knowledge on default times τ_i, $(i \in \Theta)$, together with an associated mark L_i taking values in some Polish space E. So the global market information is described by the filtration

$$\mathbb{G}^\circ = \mathbb{F}^\circ \vee \mathbb{D}^1 \vee \cdots \vee \mathbb{D}^n \vee \mathbb{D}^{L_1} \vee \cdots \vee \mathbb{D}^{L_n},$$

where for any $i \in \Theta$, the filtration $\mathbb{D}^i$ is as in Section 2.1, and the filtration $\mathbb{D}^{L_i} = (\mathcal{D}_t^{L_i})_{t\geq 0}$ is defined by $\mathcal{D}_t^{L_i} = \sigma(L_i 1\!\!1_{\{\tau_i \leq s\}},\, s \leq t)$ made right-continuous. Note that any $\mathbb{G}^\circ$-optional (resp. predictable) process Z can be written in the form

$$Z_t = \sum_{I\subset\Theta} 1\!\!1_{A_t^I} Z_t^I(\tau_I, L_I), \quad (\text{resp. } Z_t = \sum_{I\subset\Theta} 1\!\!1_{A_{t-}^I} Z_t^I(\tau_I, L_I),)$$

where $Z^I(\cdot)$ is a $\mathcal{O}_{\mathbb{F}^\circ} \otimes \mathcal{B}(\mathbb{R}_+^I \times \mathbb{R}^I)$ (resp. $\mathcal{P}_{\mathbb{F}^\circ} \otimes \mathcal{B}(\mathbb{R}_+^I \times \mathbb{R}^I)$)-measurable function, $L_I = (L_i)_{i\in I}$. In particular, the control process π can be written as

$$\pi_t = \sum_{I\subset\Theta} 1\!\!1_{A_{t-}^I} \pi_t^I(\tau_I, L_I)$$

and the wealth process X as $X_t = \sum_{I\subset\Theta} 1\!\!1_{A_t^I} X_t^I(\tau_I, L_I)$.

We explain below how to interpret the above model as a constrained optimization problem. The point is to introduce suitable auxiliary filtrations. Let $\mathbb{F} := \mathbb{F}^\circ \vee \sigma(L_1, \cdots, L_n)$. It is the initial enlargement of $\mathbb{F}^\circ$ by including the family of marks $\boldsymbol{L}$. Define also $\mathbb{F}^I := \mathbb{F}^\circ \vee \sigma(L_i, i \in I)$. Observe that the control $\pi^I(\cdot, L_I)$ is actually $\mathcal{P}_{\mathbb{F}^I} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable and is hence $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable. The wealth $X^I(\cdot, L_I)$ is $\mathcal{O}_{\mathbb{F}^I} \otimes \mathcal{B}(\mathbb{R}_+^I)$- and hence $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable. We introduce the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \vee \cdots \mathbb{D}^n$, which is the progressive enlargement of the filtration $\mathbb{F}$ with respect to the default filtrations. Note that $\mathbb{G} \supset \mathbb{G}^\circ$.

The Example 4.11 can be considered as a constrained problem by using the auxiliary filtrations $\mathbb{F}$ and $\mathbb{G}$. Indeed, an admissible control process π has now the decomposed form $\pi_t = \sum_{I\subset\Theta} 1\!\!1_{A_{t-}^I} \pi_t(\tau_I)$, where $\pi^I(s_I)$ is $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable and $\pi^I(s_I) \in \mathcal{A}^I(s_I)$, the admissible set $\mathcal{A}^I(s_I)$ being defined in Definition 4.6. Let us now make precise the constrained admissible strategy sets: let $\mathcal{A}_\circ^I(s_I)$ be the subset of $\mathcal{A}^I(s_I)$ such that $\pi^I(s_I)$ is $\mathcal{P}_{\mathbb{F}^I} \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable if $\pi^I(s_I) \in \mathcal{A}_\circ^I(s_I)$. By Remark 4.10, we can apply Theorem 4.9 to solve the problem. We finally remark that we only need the density hypothesis of $\boldsymbol{\tau}$ with respect to the filtration $\mathbb{F}$ but not necessarily the stronger one on the existence of the joint density of $(\boldsymbol{\tau}, \boldsymbol{L})$ with respect to $\mathbb{F}^\circ$.

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